Nested Elliptic Orbits in a Linear Force Field

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This work examines a system of nested elliptic orbits, in an attractive central force field where gravitational force is proportional to distance from the attracting centre of mass.

Newton expressed surprise upon finding that all orbits in this gravity field describe ellipses whose centres are at the centre of attraction, as described in reference **(1)**. Some of the further remarkable properties of orbits in this gravity field, are described in reference **(2)** where it is referred to as the harmonic force law field.

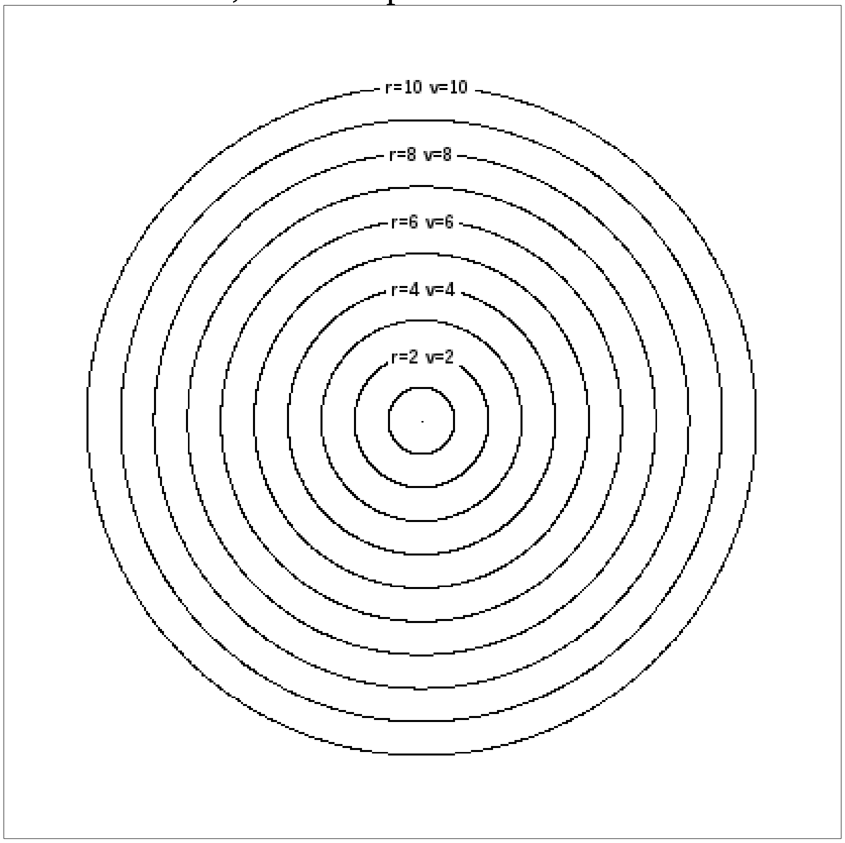
The orbits and their velocities are here examined empirically, using an n-body gravitational integration computer program, especially adapted to model the orbits of test particles of negligible mass in a centrally attractive harmonic force law field. The same results should presumably be easily obtainable analytically from harmonic oscillator theory.

The system of units used here is chosen so that a test particle in circular orbit at radius $\mathbf{r} = 1$ from the centre of attraction, has an orbital velocity $\mathbf{v_{circ}} = 1$.

The axis ratio $\mathbf{b}$ of an elliptic orbit is defined here as the semi-minor axis divided by the semi-major axis. So the axis ratio is in the range ($0 \leq \mathbf{b} \leq 1$). In the limiting case where an orbit has axis ratio $\mathbf{b} = 1$, the orbit is circular. Circular orbits have orbital velocities proportional to distance from the centre of attraction.

Figure 1 shows a system of 10 nested circular orbits. The black lines represent the orbits. The numbers represent the radii and the velocities of some of the orbits.

Figure 1: Nested circular orbits, with sample velocities

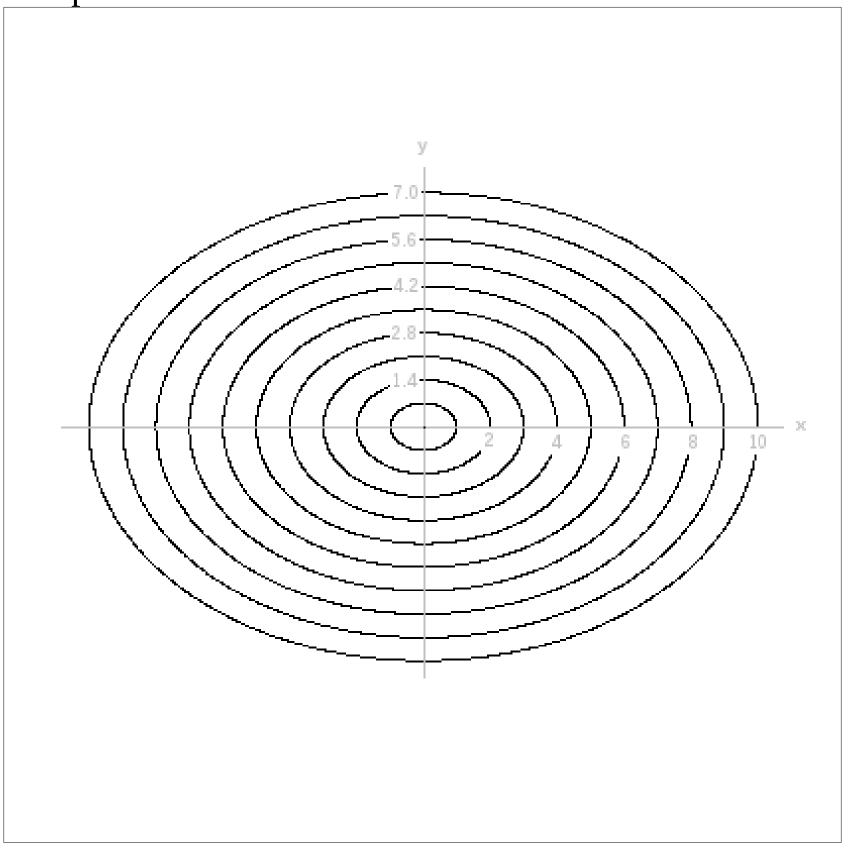


In a centrally attractive harmonic force law field, and using a system of units as defined above, the velocity of a circular orbit of orbital radius **r** is given by the simple relationship $\mathbf{v_{circ} = r}$.

Next a system of 10 nested elliptic orbits is constructed. Note that in this harmonic force law field, the centres of all orbits are at the centre of attraction.

Figure 2 shows a system of 10 nested elliptic orbits, all sharing the same axis ratio **b** = 0.7. The black lines represent the orbits. The numbers on the axes represent distances from the centre of attraction.

Figure 2: Nested elliptic orbits



A remarkable property of the centrally attractive harmonic force law field is that in a given system, all orbits have the same orbital period, independent of orbital size and ellipticity **(2)**.

Figure 3 shows 10 test particles on nested circular orbits. The 10 particles are started in alignment at time t = 0, and their progress is traced until time $t \approx 3\pi/4$. The 10 particles (red crosses) remain aligned. So in this harmonic force law field, if all the orbits are circular, then the system can be described as rotating like a solid body.

Figure 3: Nested circular orbits with same orbital period

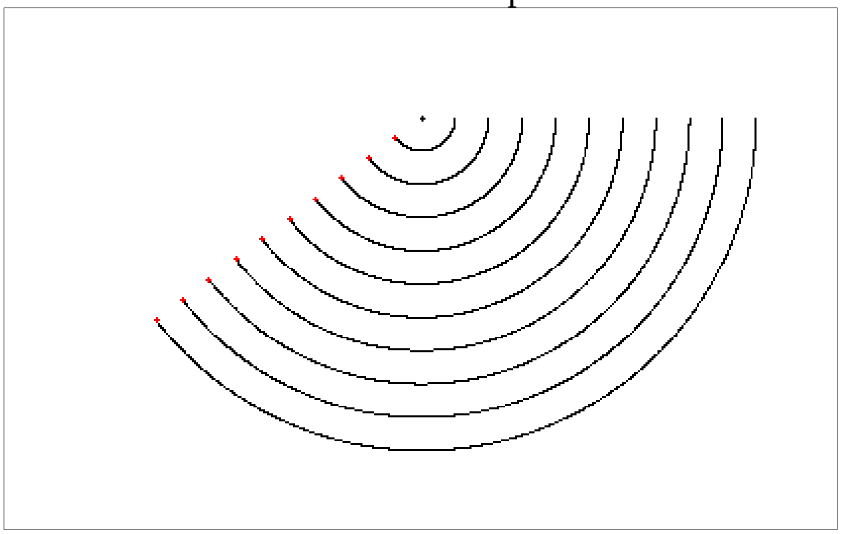
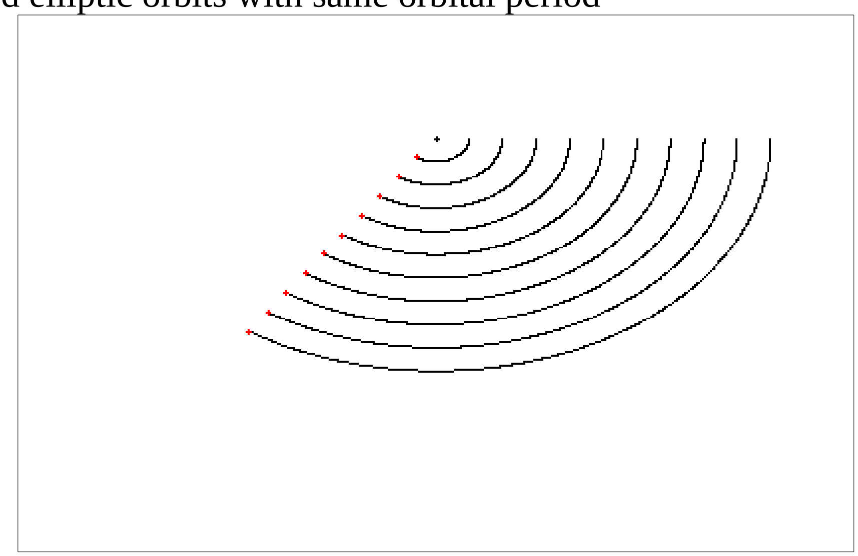


Figure 4 shows 10 test particles on nested elliptic orbits all with axis ratio **b** = 0.7. The 10 particles are started in alignment at time t = 0, and their progress is traced until time $t \approx 3\pi/4$. The 10 particles (red crosses) remain aligned. However the distance between neighboring test particles is greater at apocentre than at pericentre. So if any or all of the orbits are elliptic (with axis ratio < 1), then strictly speaking the system does not rotate exactly as a solid body. The orbits in figures 3 and 4 all have the same orbital period.

Figure 4: Nested elliptic orbits with same orbital period

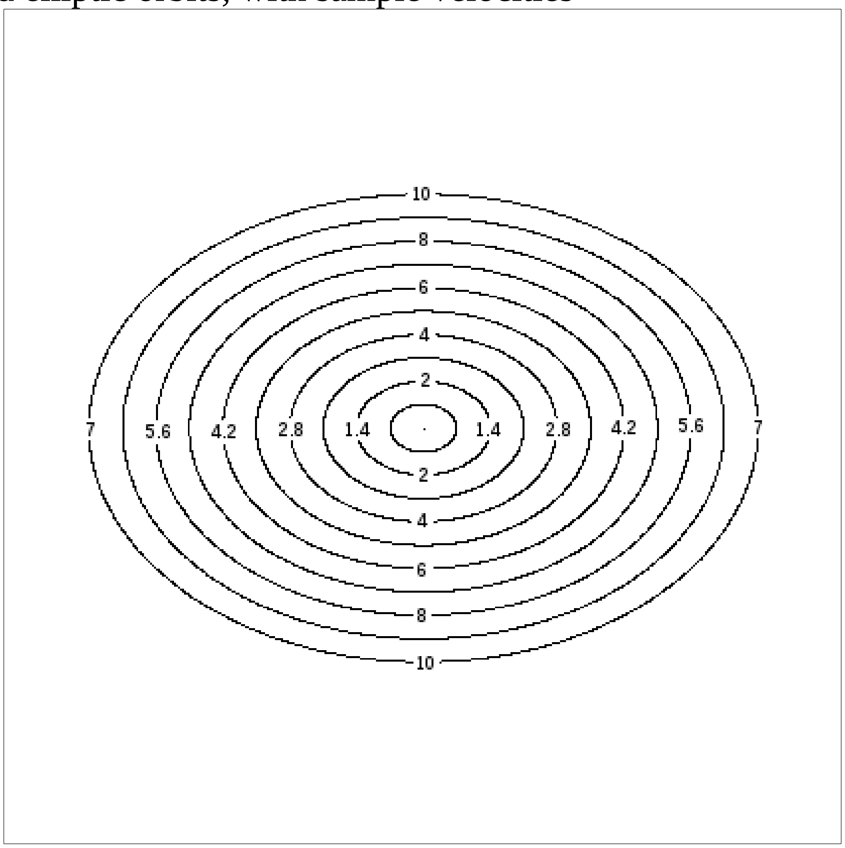


For an elliptic orbit in a centrally attractive harmonic force law field, the orbital velocities at pericentre and at apocentre can be calculated using the method described in reference **(3)** as follows:

Two circular orbits are constructed which exactly enclose the elliptic orbit.
The orbital velocity of the elliptic orbit at pericentre (*innermost* point) is equal to the orbital velocity of the *outer* circular orbit. The orbital velocity of the elliptic orbit at apocentre (*outermost* point) is equal to the orbital velocity of the *inner* circular orbit.

Figure 5 shows the system of 10 elliptic orbits with axis ratio **b** = 0.7. The velocities at pericentre and at apocentre of each of the 10 elliptic orbits were calculated by the above method. The lines represent the orbits. The numbers represent sample velocities at pericentre and apocentre of some of the orbits.

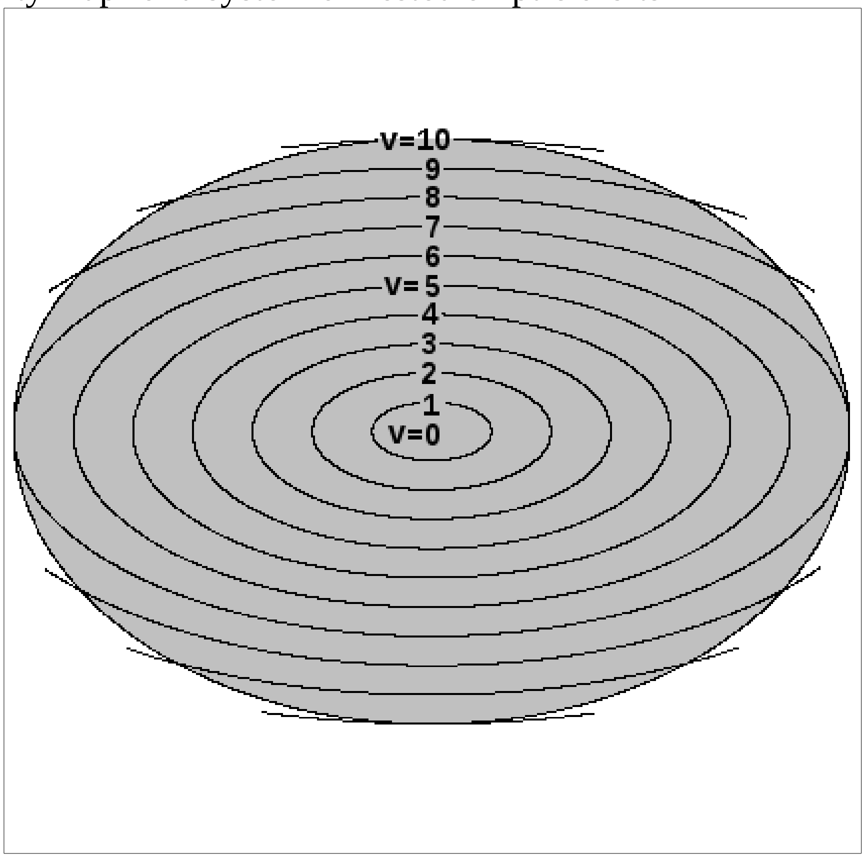
Figure 5: Nested elliptic orbits, with sample velocities



Using a larger sample of velocity values, a velocity map was generated. Each line of the velocity map is a contour. All points on a given contour have equal velocity.

Figure 6 shows the velocity map for the system of 10 elliptic orbits with axis ratio **b** = 0.7. The grey shaded area represents the system of orbits, bounded by a black ellipse which represents the outermost orbit (the inner 9 orbits are not shown). The remaining black lines are isovelocity contours. Some of the the contours are truncated where they extend beyond the outer orbit of the system. The numbers are the velocity values for each of the isovelocity contours.

Figure 6: Velocity map for a system of nested elliptic orbits

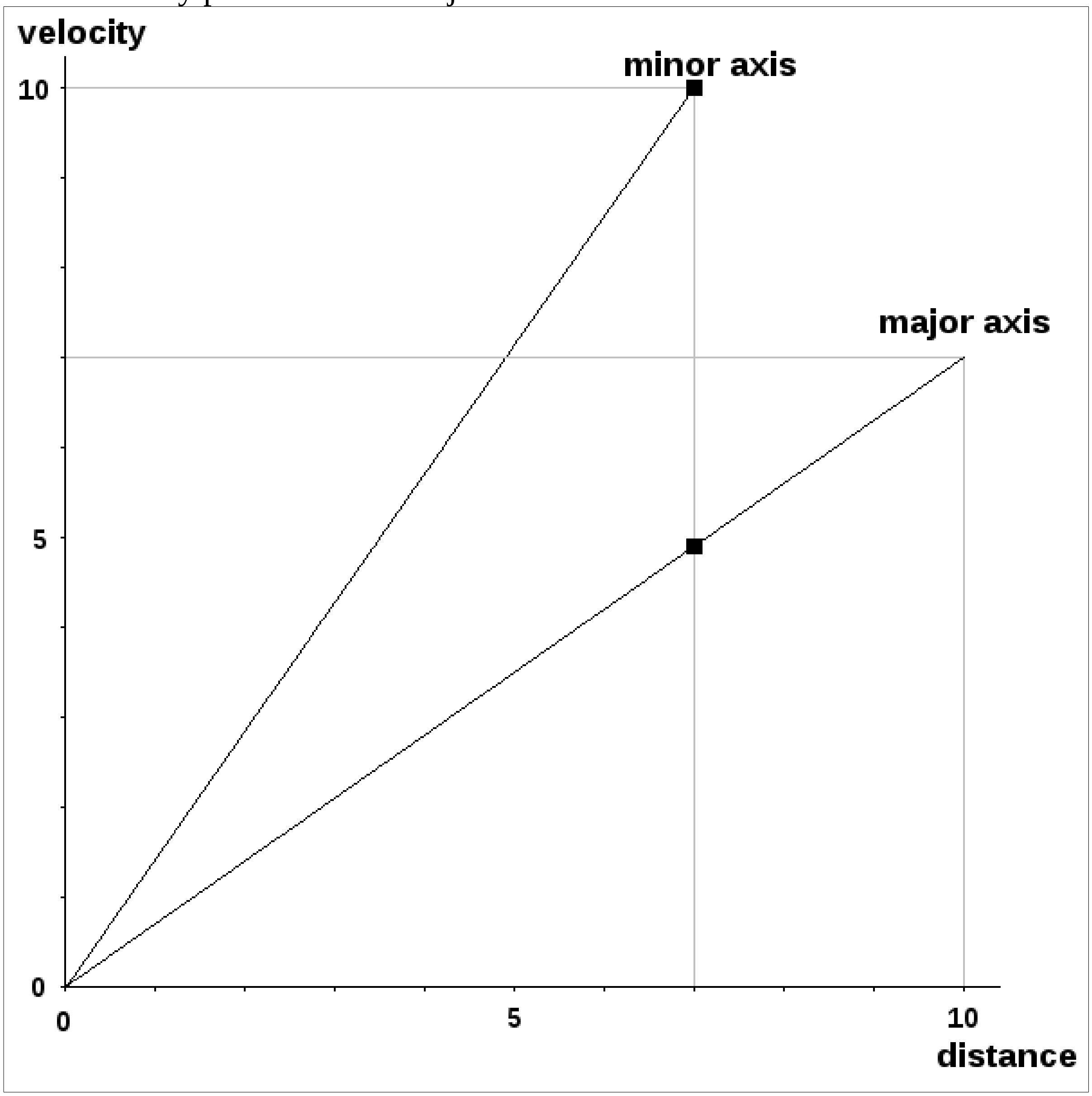


The isovelocity contours are described by a system of nested ellipses, whose major axes are aligned with the major axes of the orbit ellipses. However, the isovelocity contours have a different axis ratio. The isovelocity contours have axis ratio **= $(0.7)^2$ = 0.49**.

Conclusion 1:
In a centrally attractive harmonic force law field, a system of nested elliptic orbits, all with orbital axis ratio = **b**, will have isovelocity contours described by a seperate system of nested ellipses all with isovelocity contour axis ratio = $\mathbf{b}^2$.

Figure 7 shows the velocity profiles, measured on the major axis, and measured on the minor axis, for the system of nested elliptic orbits with axis ratio **b** = 0.7.

Figure 7: Velocity profiles on the major axis and minor axis



Here the velocity ratio is defined as: the velocity measured at distance **d** on the major axis, divided by the velocity measured at distance **d** on the minor axis.

In figure 7, the two black dots illustrate that, at **d** = 7:
The velocity measured on the minor axis = **10**,
The velocity measured on the major axis = $(0.7)^2$ x 10 = **4.9**.
Therefore the **velocity ratio = $(0.7)^2$ = 0.49**.

Conclusion 2:
In a centrally attractive harmonic force law field, a system of nested elliptic orbits, all with orbital axis ratio = **b**, will have major axis and minor axis velocity profiles related by the **velocity ratio = b^2**.

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