The Velocity of a Centred Elliptical Orbit

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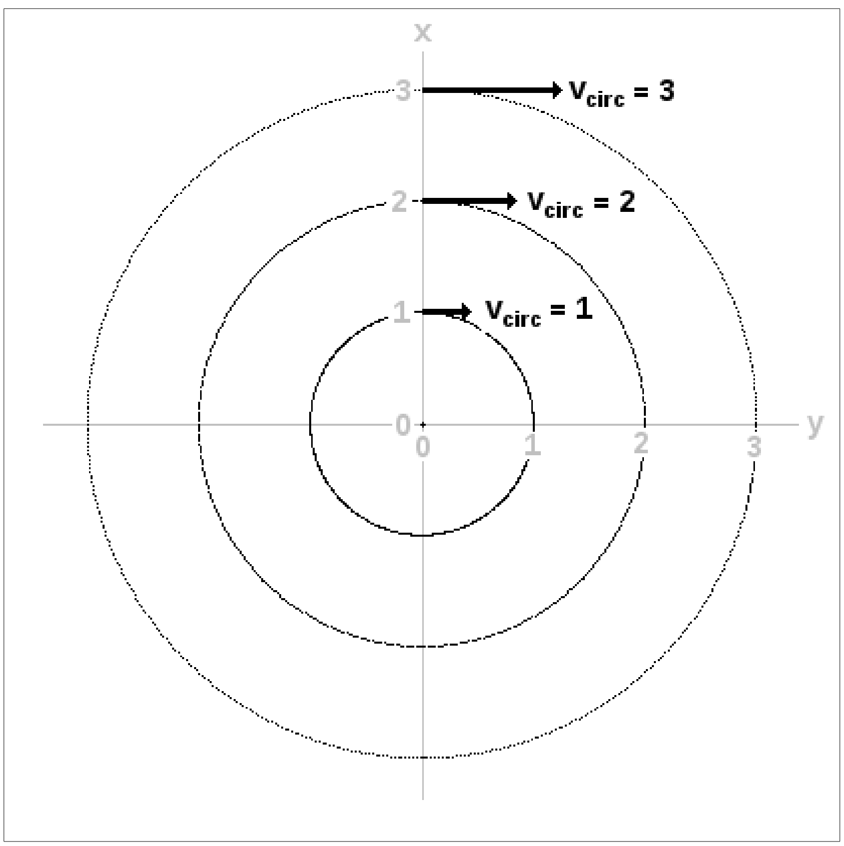
In an attractive central gravity field, in which the gravitational attraction is proportional to the distance from the centre of attraction, the orbit of a test particle describes an ellipse whose centre is at the centre of attraction.
That result was deduced by Newton. Further properties of orbits in this gravity field have been examined by various authors, and are concisely listed by Blitzer (1). Most remarkable is that in a system with this gravity field, all orbits (with various semi-major-axes and various axis ratios) will have the same orbital period.

The aim of this work is to examine the velocity, at pericentre and at apocentre, of a centred elliptical orbit in this gravity field. The system of units is chosen so that a circular orbit, at distance **d** = 1 from the centre of attraction, has an orbital velocity $v_{circ} = 1$.
For orbits with axis ratio = 1 (circular orbits), the orbital velocity, in this gravity field, is proportional to the radius of the orbit. It is noteworthy that larger circular orbits have greater velocities than smaller circular orbits. Figure 1 shows three circular orbits and their orbital velocities.

Figure 1

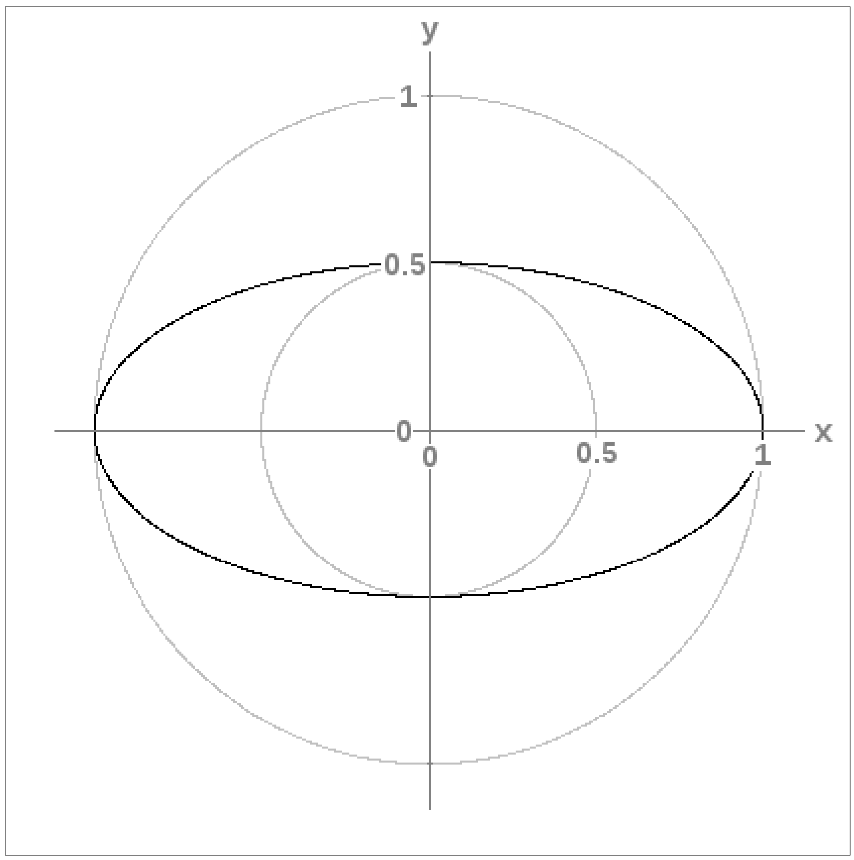


The axis ratio of an elliptical orbit is defined here as the ratio of the semi-minor-axis to the semi-major axis, and thus is always within the range ($0 \leq$ axis ratio ≤ 1).

For orbits with axis ratio < 1 (non-circular elliptical orbits), the orbital velocity of an test particle varies as it passes around its orbit.

Figure 2 shows a centred elliptical orbit with axis ratio = 0.5. Also shown are two circular orbits which exactly enclose the elliptical orbit.

Figure 2



The orbital velocities of a centred elliptical orbit, at pericentre and at apocentre, have the following relationship with the orbital velocities of the two circular orbits which enclose it:

(a) The orbital velocity of the centred elliptical orbit at pericentre (*innermost* point) is equal to the orbital velocity of the *outer* circular orbit.

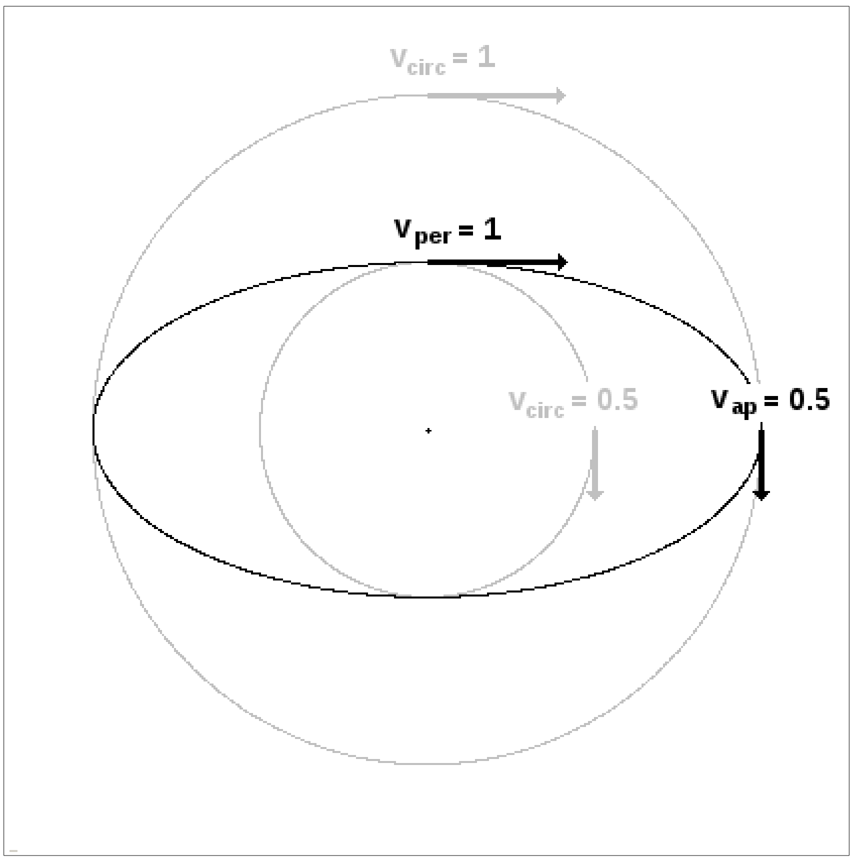
Vper = Vcirc(outer)

(b) The orbital velocity of the centred elliptical orbit at apocentre (*outermost* point) is equal to the orbital velocity of the *inner* circular orbit.

Vap = Vcirc(inner)

The outer circular orbit in figure 2 has radius = 1 and orbital velocity **Vcirc(outer)** = 1.
The inner circular orbit has radius = 0.5 and therefore has orbital velocity **Vcirc(inner)** = 0.5.
Therefore this elliptical orbit has orbital velocity at pericentre **Vper = Vcirc(outer)** = **1**,
and orbital velocity at apocentre **Vap** = **Vcirc(inner)** = **0.5**,
as illustrated in figure 3.

Figure 3



Here is another example, again using the system of units defined above. Consider a centred elliptical orbit with semi-major axis = 1, and semi-minor axis = 0.1. The orbital velocities at pericentre and at apocentre are calculated from the velocities of the two enclosing circular orbits as follows:

The outer circular orbit has radius = 1 and orbital velocity $\mathbf{V_{circ(outer)}}$ = 1.

The inner circular orbit has radius = 0.1 and therefore has orbital velocity $\mathbf{V_{circ(inner)}}$ = 0.1.

So the orbital velocity at pericentre $\mathbf{V_{per}}$ = $\mathbf{V_{circ(outer)}}$ = **1**

and the orbital velocity at apocentre $\mathbf{V_{ap}}$ = $\mathbf{V_{circ(inner)}}$ = **0.1**

Generically, in a gravity field in which gravitational attraction towards the centre is proportional to distance from the centre, an elliptical orbit with axis ratio (semi-minor-axis / semi-major axis) = **b,** will have a velocity ratio (velocity at apocentre / velocity at pericentre) = **b**.

These results are not new, and may be obtained empirically by computer simulation, or analytically from harmonic oscillator theory.

In part two of this work, the simple relationship described above will be applied to investigate the velocity map of a system of aligned and nested centred elliptical orbits.

References

[1] Blitzer, L.
Hyper-elliptic orbits
Celestial Mechanics 42 (1988) 215-222
http://adsabs.harvard.edu/full/1988CeMec..42..215B